Schrödinger, E. 1939, Physica, 6, 899

THE PROPER VIBRATIONS OF THE EXPANDING UNIVERSE

by ERWIN SCHRÖDINGER

§ 1. *Introduction and summary*. Wave mechanics imposes an a priori reason for assuming space to be closed; for then and only then are its proper modes discontinuous and provide an adequate description of the observed atomicity of matter and light. — Einsteins theory of gravitation imposes an a priori reason for assuming space to be, if closed, expanding or contracting; for this theory does not admit of a stable static solution. — The observed facts are, to say the least, not contrary to these assumptions.

This makes it imperative to generalize to expanding (or contracting) universes the investigation of proper vibrations, started for the the static cases (Einstein- and De Sitter-universe) by the present writer and two of his collaborators [1]). The task is an easy one. The broad results are largely (in part even entirely) independent of the time-law of expansion. In the cases of main practical interest, i.e. with the present slow time rate of expansion and with wave lengths small compared with the radius of curvature of space (R), they are the following.

For *light*: when referred to the customary *co-moving* coordinates, an *arbitrary* wave process exhibits essentially the same succession of states as without expansion. Briefly, the wave function shares the general dilatation. Hence all *wave lengths* increase proportionally to the radius of curvature. — The *time rate* of events is slowed down. It is, in every moment, proportional to R^{-1}. Moreover all *intensities* are affected by a common factor such as to make the total energy of an arbitrary wave process proportional to R^{-1}.

For the *material particle* the broad results are these: a strictly monochromatic process (i.e. a proper vibration) again shares the common dilatation, so that its wave length λ is proportional to R, as before. From the changing λ the changing *frequency* is calculated by de Broglies formula. This implies different frequencies to be affected by different factors. Therefore an arbitrary wave function can no longer be said to simply share the common dilatation. But since de Broglie's dispersion formula persists, the familiar connection (momentum $= h/\lambda$) between linear group velocity ($=$ particle velocity) and wave length is also preserved, which causes the former or more precisely the momentum, to decrease proportional to R^{-1}. As regards the amplitudes, the most reliable information about them, valid for any particle wave function whatsoever, is this, that the *normalisation* is rigorously conserved during the expansion.

These are the broad results. A finer and particularly interesting phenomenon is the following.

The decomposition of an arbitrary wave function into proper vibrations is rigorous, as far as the functions of space (amplitude-functions) are concerned, which, by the way, are exactly the same as in the static universe. But it is known, that, with the latter, two frequencies, equal but of opposite sign, belong to every space function. *These two* proper vibrations cannot be rigorously separated in the expanding universe. That means to say, that if in a certain moment only one of them is present, the other one can turn up in the course of time.

Generally speaking this is a phenomenon of outstanding importance. With particles it would mean production or anihilation of matter, merely by the expansion, whereas with light there would be a production of light travelling in the opposite direction, thus a sort of reflexion of light in homogeneous space. Alarmed by these prospects, I have investigated the question in more detail. Fortunately the equations admit of a solution by familiar functions, if R is a *linear* function of time. It turns out, that in this case the alarming phenomena do not occur, even within arbitrarily long periods of time. Waves travelling in one direction can be rigorously separated from those travelling in the opposite direction. The results for D'Alemberts equation (light) and Gordons equation (material particles), which have been used throughout in this paper for the sake of simplicity, are given in sect. 5 and 6 respectively. I have confirmed the results with Diracs equation, but reserve it to a subsequent paper.

For all I have found hithertoo I would conclude, that the alarming phenomena (i.e. pair production and reflexion of light in space) are not connected with the *velocity* of expansion, but would probably be caused by *accelerated* expansion. They may play an important part in the critical periods of cosmology, when expansion changes to contraction or vice-versa.

§ 2. *The wave equation, its conservation theorem, its general solution.* The familiar wave equation of the second order

$$-\Delta\psi + \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \mu^2\psi = 0 \qquad (1)$$

($\mu = 0$ for light

$\mu = 2\pi mc/h$ for material particles)

is to be regarded as the covariant equation

$$g^{\alpha\beta}\psi;\alpha;\beta + \mu^2\psi \equiv \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x_\alpha}\left(g^{\alpha\beta}\sqrt{-g}\,\frac{\partial\psi}{\partial x_\beta}\right) + \mu^2\psi = 0, \qquad (2)$$

specialized for the line element

$$ds^2 = g_{\alpha\beta}dx_\alpha dx_\beta = -dx_1^2 - dx_2^2 - dx_3^2 + c^2dt^2. \qquad (3)$$

The line element of the non-static universe can be written [2])

$$ds^2 = -R^2[d\chi^2 + \sin^2\chi(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)] + c^2dt^2. \qquad (4)$$

$R(t)$ is the radius of spatial curvature at time t, the function being left open. c is a constant. χ, ϑ, φ are the well-known *co-moving* angular coordinates, they are constant for a nebula without peculiar motion. With (3) 4 equ. (2) reads

$$-R^{-2}K[\psi] + \frac{1}{c^2}R^{-3}\frac{\partial}{\partial t}\left(R^3\frac{\partial\psi}{\partial t}\right) + \mu^2\psi = 0. \qquad (5)$$

$K[\ldots.]$ is the differential operator of the second order of which the eigenfunctions are the spherical harmonics, generalized to three dimensions *). It is self-adjoint, with the density function $\sin^2\chi\sin\vartheta$. Its eigenvalues are $-n(n+2)$, with $n = 0, 1, 2, 3, \ldots..$

Equ. (5) admits of a genuine conservation-law. Multiply its left by

*) See A.P. p. 323, equ. (2. 3).

$\psi^* \sin^2\chi \sin\vartheta\, R^3 c^2\, d\vartheta d\varphi d\chi$, from the result subtract its complex conjugate and integrate over the whole space. You get

$$\frac{d}{dt}\iiint\left(\psi^*\frac{\partial\psi}{\partial t}-\psi\frac{\partial\psi^*}{\partial t}\right)R^3\sin^2\chi\sin\vartheta\, d\vartheta d\varphi d\chi = 0. \qquad (6)$$

The bracket-expression is just what, with the force-free G o r d o n-equation, corresponds to the density of probability (or electricity). Thus for an arbitrary material wave function the ***normalization*** is strictly conserved during the expansion. (I have confirmed this result also for D i r a c's equation). In the case of *light* ψ is, properly speaking, real and equ. (6) becomes, properly speaking, trivial.

The general solution of (5) is accomplished by the classical method of separation of variables. Put

$$\psi(\chi, \vartheta, \varphi, t) = \omega(\chi, \vartheta, \varphi)\, f(t), \qquad (7)$$

ω being an eigenfunction of K, known from A.P. For $f(t)$ you obtain

$$R^{-3}\frac{d}{dt}\left(R^3\frac{df}{dt}\right)+\frac{c^2 n(n+2)}{R^2}f + c^2\mu^2 f = 0. \qquad (8)$$

Take in (7) for $f(t)$ a linear aggregate, formed of two independent solutions of (8) with the help of two arbitrary constants. Form an infinite of all the solutions like (7). The series can, in the familiar way, be adapted to an arbitrary initial state. What becomes of one of its members in the course of time is independent from all the rest. If at the outset only one was present, that will remain so. We are thus in face of a genuine decomposition into proper vibrations, although the time-factors $f(t)$ are in general not trigonometric functions. They would, of course, assume and re-assume this form in the moment when and as often as $R(t)$ would cease to vary and would remain constant for a time, and during such time every proper vibration would asssume the frequency due to it in that static universe. For light ($\mu = 0$) all these frequencies are inversely proportional to R.

We have quite intentionally called *one* proper vibration the term containing *one* particular spatial function ω, but *both* solutions of (8). The latter correspond to what with $R =$ Const. would be $\cos 2\pi\nu t$ and $\sin 2\pi\nu t$; or, alternatively to $e^{2\pi i\nu t}$ and $e^{-2\pi i\nu t}$. Of course the two parts keep clear of each other also in the general case. But for assigning a quite general physical meaning to this separation, one would have to know, that an $f(t)$ which during a period of constant R (or very slowly varying R) had the form (or approximately the form) $e^{2\pi i\nu t}$ will re-assume (or approximately re-assume) the form $Ae^{2\pi i\nu' t}$ — and *not* $Ae^{2\pi i\nu' t} + Be^{-2\pi i\nu' t}$ — whenever $R(t)$, after an intermediate period of arbitrary variation, returns to constancy (or to approximate constancy). I can see no reason whatsoever for $f(t)$ to behave rigorously in this way, and indeed I do not think it does. There will thus be a mutual adulteration of positive and negative frequency terms in the course of time, giving rise to what in the introduction I called „the alarming phenomena". They are certainly very slight, though, in two cases, viz. 1) when R varies slowly 2) when it is a linear function of time (see the following sections).

A second remark about the new concept of proper vibration is, that it is not always invariantly determined by the form of the universe. The separation of time from the spatial coordinates may succeed in a number of different space-time-frames. For D e S i t t e r s universe I know three of them. Besides the static one, for which P. O. M ü l l e r (l.c.) has redently given the proper vibrations, there is an expanding form with infinite R and an expanding form with finite R *). A proper vibration of one frame will not transform into a proper vibration of the other frame, for the separation of variables is destroyed by the transformation.

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*) From D e S i t t e r s line-element in static form

$$ds^2 = -R_0^2[d\chi^2 + \sin^2\chi(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)] + R_0^2\cos^2\chi\, dt^2$$

the transformation of L e m a î t r e (J. Math. and Phys. M.I.T., 4, 188, 1925) and R o b e r t s o n (Phil. Mag. 5, 835, 1928)

$$\bar r = R_0 \operatorname{tg}\chi\, e^{-t} \qquad \bar t = t + \lg\cos\chi$$

gives the expanding *flat* form

$$ds^2 = -e^{2\bar t}[d\bar r^2 + \bar r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)] + R_0^2\, d\bar t^2.$$

The following transformation

$$\operatorname{tg}\chi' = \frac{\operatorname{tg}\chi}{\operatorname{Cos} t} \qquad \operatorname{Sin} t' = \operatorname{Sin} t\cos\chi$$

or

$$\sin\chi = \sin\chi' \operatorname{Cos} t' \qquad \operatorname{Tg} t = \operatorname{Tg} t'(\cos\chi')^{-1}$$

gives the expanding *curved* form

$$ds^2 = -R_0^2(\operatorname{Cos} t')^2[d\chi'^2 + \sin^2\chi'(d\vartheta^2 + \sin^2\vartheta d\varphi^2)] + R_0^2\, dt'^2.$$

(In this footnote R_0 is a constant length and the cosmical times t, $\bar t$, t' are dimensionless.)

§ 3. *The secular variation of amplitudes.* In equ. (8) introduce a new independent variable τ by

$$d\tau = R^{-3}dt, \tag{9}$$

giving you

$$\frac{d^2f}{d\tau^2} = -[c^2n(n+2)R^4 + c^2\mu^2R^6]f. \tag{10}$$

This is the equation of a pendulum with slowly varying constants. The varying frequency is

$$\nu' = \frac{R^3c}{2\pi}\sqrt{\frac{n(n+2)}{R^2} + \mu^2}. \tag{11}$$

The laws of adiabatic transformation will apply, provided $R^{-1}\,dR/d\tau$ is small compared with ν', or

$$R^{-1}dR/dt \ll \frac{\nu'}{R^3} = \frac{c}{2\pi}\sqrt{\frac{n(n+2)}{R^2} + \mu^2} = \nu, \tag{12}$$

say. In the cases of practical interest this is amply fulfilled, for $2\pi R/n$ is the wave length, hence n is a very large number and ν is, by the last equation, the *true* frequency both in the case of light ($\mu = 0$) and in the case of De Broglie waves ($\mu = 2\pi mc/h$). Following Ehrenfests law of adiabatic transformation the energy of the pendulum will exhibit a secular variation proportional to ν'. This means $\nu'^2 f^2 \sim \nu'$ or

$$f^2 \sim \frac{1}{\nu'} = \frac{1}{\nu R^3}. \tag{13}$$

This is immediately applicable only, when f is *real*, as it is with a pendulum.

As a first application, consider the most general complex solution of (10), which is certainly of the form

$$f = Ae^{2\pi i\nu t} + Be^{-2\pi i\nu t}, \tag{14}$$

with A, B and ν varying slowly with time, the latter according to the last equation (12). Since (10) has real coefficients, the real and the imaginary part of (14) are themselves solutions and we can apply (13) to *them*. This gives by a simple calculation

$$(|A|^2 + |B|^2) \sim \frac{1}{\nu R^3}. \tag{15}$$

On the other hand we can apply our conservation theorem from § 2 to (14) and obtain *)

$$(|A|^2 - |B|^2) \sim \frac{1}{\nu R^3}. \tag{16}$$

From (15) and (16) follows, that $|A|^2$ and $|B|^2$ themselves follow the same law. Therefore if e.g. B was initially zero, it will remain zero. We have the important result:

To the degree of approximation of Ehrenfests theorem there is no mutual contamination of positive and negative frequency solutions.

A second application is to the energy density of light. With D'Alemberts equation it is proportional to $\nu^2 f^2$, therefore we have

$$\text{energy density} \sim \frac{\nu}{R^3}, \tag{17}$$

which gives the total energy of a proper vibration proportional to ν or to R^{-1}.

If one choses to speak of an energy density of material waves, the law (17), i.e. $\sim \nu/R^3$, also holds for it — to the degree of approximation of Ehrenfests theorem. But there is no point in that, since the conservation of normalization in this case gives more complete and rigorous information.

§ 4. *Group velocity.* We now turn to investigate the most important feature arising from the *superposition* of different proper vibrations. Since everyone of them will show a *secular phase-shift*, we have to investigate, whether or to what extent this might interfere with the fine interlocking of phases that produces *group-velocity*.

Assume a solution of (10) in theform

$$f_n = A_n e^{-i\vartheta_n} \tag{18}$$

with real A_n and ϑ_n, the first varying slowly, the second approximately linearly with τ, i.e. with two coefficients that vary slowly. (We have proved in the preceding section that these assumptions are

*) A more direct way of proving (16) is to apply to the real and to the imaginary part of (14) the relation

$$f_1\frac{df_2}{d\tau} - f_2\frac{df_1}{d\tau} = \text{Const.}$$

which holds for any two solutions f_1 and f_2 of equ. (10).

legitimate). The subscript n refers to the integer occurring in (10). An appropriate space function ω, to produce with f_n a progressive wave along a great circle is $e^{in\phi}$ (see A.P. p. 328, equ. 3.3; the factors containing ϑ and χ are immaterial here). So we contemplate

$$\psi_n = f_n e^{in\phi} = A_n e^{i(n\phi - \vartheta_n)}.$$

By equating to zero the differential of the exponent, we find the *phase-velocity* c'_{ph}, which, for the moment, we shall measure as an *angular* velocity and with respect to the variable τ. Thus

$$c'_{ph} = \frac{1}{n}\frac{d\vartheta_n}{d\tau} = \frac{i}{n}\frac{d\,lg\,f_n}{d\tau}. \tag{19}$$

The *last* equation holds with neglect of the variation of the *amplitude* A_n (already known to vary very slowly).

The *group-velocity* c'_{gr} (again angular and with respect to τ) is obtained by equating to zero the *second* differential of the phase, taken with respect to both τ and n. We get

$$c'_{gr} = \frac{d\,\Delta\,\vartheta_n}{d\tau} = i\,\frac{d\,\Delta\,lg\,f_n}{d\tau}. \tag{20}$$

The sign Δ means quasi-differentiation with respect to the integer n and the *last* equation is even safer than in (19), since we may chose A_n initially independent of n.

Now make in equation (10) the Riccati transformation

$$y = \frac{d\,lg\,f}{d\tau}, \tag{21}$$

which turns it into

$$\frac{dy}{d\tau} + y^2 + c^2 n(n+2)R^4 + c^2\mu^2 R^6 = 0. \tag{22}$$

„Differentiate" this with respect to n:

$$\frac{d\,\Delta y}{d\tau} + 2y\,\Delta y + 2c^2(n+1)R^4 = 0. \tag{23}$$

From (19), (20) and (21)

$$y = -inc'_{ph}, \qquad \Delta y = -ic'_{gr}.$$

Neglecting the *variation* of group velocity with τ, we get from (23)

$$c'_{ph}c'_{gr} = \frac{c^2(n+1)}{n}R^4,$$

or, for the *true* and *linear* velocities

$$c_{gr}c_{ph} = \frac{n+1}{n}c^2. \tag{24}$$

Since n is extremely large, this is the familiar relation, valid for both light and De Broglie waves. Since from the last equ. (12) or, alternatively, from (22) the familiar value is easily deduced for c_{ph}, the modification of c_{gr} is likewise unappreciable. Quantitative results will be obtained in the following sections.

§ 5. *Closed solution for light, when the radius is a linear function of time.* In this and the following section we investigate the special case

$$R = a + bt. \tag{25}$$

Following (9) we put

$$\tau = \int_{-\infty}^{t}\frac{dt}{(a+bt)^3} = -\frac{1}{2b(a+bt)^2} = -\frac{1}{2bR^2}. \tag{26}$$

Hence from (10)

$$\frac{df^2}{d\tau^2} + \left(\frac{c^2n(n+2)}{4b^2\tau^2} - \frac{c^2\mu^2}{8b^3\tau^3}\right)f = 0. \tag{27}$$

Specialising for light ($\mu = 0$) and putting for the moment

$$\frac{c^2n(n+2)}{b^2} = k^2 + 1, \tag{28}$$

we have

$$\frac{d^2f}{d\tau^2} + \frac{k^2+1}{4\tau^2}f = 0,$$

of which the solutions are

$$f = \tau^\beta \text{ with } \beta = \tfrac{1}{2} \pm \tfrac{1}{2}ik$$

thus

$$\left.\begin{aligned} f &= \frac{1}{a+bt}(a+bt)^{\pm ik} \\ &= \frac{1}{R}R^{\pm ik} \end{aligned}\right\} \tag{29}$$

Since n is very large, k is real, for b is certainly not much larger than c. Hence the second factor has absolute value 1 and is the oscillating part, whereas the first factor is the amplitude, which in this particular case is seen to be *exactly* proportional to R^{-1}.

Two main inferences can be drawn from the solutions (29). If we treat them as in the preceding section we treated (18), combining them with the space-function $e^{in\phi}$, we can write

$$\psi = fe^{in\phi} = \frac{1}{a + bt} e^{i[n\phi \pm \sqrt{(c^2 n(n+2))/b^2 - 1}\,.\,\lg(a+bt)]}. \tag{30}$$

These two progressive waves, travelling in opposite directions, are, in the present case, rigorous solutions and will therefore keep rigorously separated for any length of time. No doubt they are not true exponential waves. If the linear expansion only sets in in a certain moment and comes to rest in a later moment, then *in these two moments* there may be a small amount of contamination.

Next, we calculate the accurate values of the phase- and group-velocity from (30). Proceeding exactly as in the preceding section we find:

$$\left.\begin{aligned} c_{ph} &= \frac{c}{n}\left((n(n+2) - \frac{b^2}{c^2}\right)^{\frac{1}{2}} \\ c_{gr} &= c(n+1)\left(n(n+2) - \frac{b^2}{c^2}\right)^{-\frac{1}{2}} \end{aligned}\right\} \tag{31}$$

Thus c_{ph} is slightly smaller, c_{gr} is slightly greater than without expansion. But as long as b/c is of the order of unity, the effect does not exceed that of curvature itself, viz. it is extremely small.

§ 6. *The same for material waves.* The variable τ is no longer convenient. We therefore return to (8), make the assumption (25) and introduce in (8) the new independent variable

$$z = \frac{\mu c R}{b} = \frac{\mu c a}{b} + \mu c t \tag{32}$$

and the new dependent variable

$$w(z) = zf, \tag{33}$$

which turns (8) into

$$\frac{d^2 w}{dz^2} + \frac{1}{z}\frac{dw}{dz} + \left(1 + \frac{k^2}{z^2}\right) w = 0 \tag{34}$$

(k^2 is the same as in (28)). So w is a Bessel function of the purely imaginary order ik. On inspection it is seen, that both k and z are enormously great, whereas z/k is of the comparatively moderate order: actual wave length divided by Compton wave length. This is the proper working ground for the method of steepest descent, introduced by P. Debye *) into this branch of analysis; it has only to be adapted to the present case of imaginary order. Let us consider the first kind Hankel function

$$H^1_{ik}(z) = -\frac{1}{\pi}\int e^{-iz\sin\zeta - k\zeta}\, d\zeta, \tag{35}$$

the path of integration being primarily $-i\infty \to 0 \to -\pi \to -\pi + i\infty$. I find the suitable point of steepest descent to be $\zeta = -\pi/2 + i\alpha$ with

$$\operatorname{Sin}\alpha = \frac{k}{z} \tag{36}$$

and the appopriate deformed path of integration to ascend in this point from right to left under 45°. My result is

$$H^1_{ik}(z) = \frac{(1-i)e^{k\pi/2}}{\sqrt{\pi z \operatorname{Cos}\alpha}}\, e^{ik(\operatorname{Cotg}\alpha - \alpha)}. \tag{37}$$

Thus from (33), if we drop an irrelevant constant multiplier,

$$f(t) = z^{-3/2}(\operatorname{Cos}\alpha)^{-1/2} e^{ik(\operatorname{Cotg}\alpha - \alpha)}. \tag{38}$$

In order to find the *frequency* (first in the very small time unit in which z is the time) we differentiate the phase with respect to z, or rather $2\pi z$; from (36) we have

$$\frac{1}{2\pi}\frac{d}{dz}[k(\operatorname{Cotg}\alpha - \alpha)] = \frac{1}{2\pi}\operatorname{Cos}\alpha. \tag{39}$$

Thus we see, by the way, that the factor preceding the exponential in (38) takes proper care of our conservation theorem, for z is proportional to R, by (32). — The *true* frequency is

$$\nu = \frac{\mu c}{2\pi}\operatorname{Cos}\alpha = \frac{c}{2\pi}\sqrt{\frac{n(n+2) - (b/c)^2}{R^2} + \mu^2}, \tag{40}$$

which is De Broglies dispersion formula, including the slight correction for the finite rate of expansion b (compare with the value (12), obtained for infinitely slow expansion). We make an explicit

*) See e.g. Courant-Hilbert, Methoden der Matematischen Physik I, 2. Auflage (Berlin, Springer 1931), S. 455 ff.

note of the phase-velocity

$$c_{ph} = \frac{2\pi R\nu}{n} = \frac{\mu c R}{n} \operatorname{Cos} \alpha \tag{41}$$

and evaluate the group-velocity

$$c_{gr} = R\mu c \frac{d \operatorname{Cos} \alpha}{dn} = \frac{c(n+1)}{\mu R \operatorname{Cos} \alpha}. \tag{42}$$

From the last two formulae

$$c_{ph} c_{gr} = \frac{n+1}{n} c^2, \tag{43}$$

which is in so *literal* agrement with (24) as could not have been anticipated.

H_{ik}^2 can. of course, be worked out in the same way and gives the exponential with the negative frequency. So here too, as in the case of light, the positive and negative frequency solutions, when properly defined, keep clear of each-other; there is nothing like a secularly accumulated pair production — at any rate not to the degree of approximation of our asymptotic formulae, which is an *extremely* high one.

§ 7. *Re-stating briefly several useful formulae.* By a few examples I wish to show, that the broad features of the wave-aspect simplify the understanding of the expanding universe.

The nebular red-shift is directly visualisable as the dilatation of all wave-lengths along with R. It is a thing that happens to every portion of light *on its journey*, along with a dilatation of all its dimensions and with a reduction of its total energy; all this is entirely independent of the origin of that portion of light. To speak of a D o p p l e r-effect is rather inappropriate, for the thing has nothing to do with dR/dt in the moment of emission or in the moment of observation, but only with the ratio of the R's of these two moments.

Moreover to the wave-aspect the slowing down of freely moving particles is on the same footing as the red-shift of light, it is just the red-shift of D e B r o g l i e waves. Only, since with a particle not the energy but the momentum varies like λ^{-1}, it is here the momentum that goes with R^{-1}; thus, for slow particles, the velocity; their energy then with R^{-2} or with $V^{-2/3}$, if V is the volume. Hence for an ideal monoatomic gas, filling the universe, we should have

$$pV \sim V^{-2/3} \text{ or } pV^{5/3} = \text{const.}$$

showing, that it behaves as on adiabatic expansion.

In certain considerations [3]) the observed angular diameter of a distant object (nebula) and its observed luminosity are of importance. Draw from the origin ($\chi = 0$) two geodesics of space to the ends of a line element l (linear diameter of the nebula), situated at a distance χ, oriented in the direction of increasing ϑ. From the expression (4) of the line-element the angle $d\vartheta$ between the geodesics is

$$d\vartheta = \frac{l}{R \sin \chi}. \tag{44}$$

If *in the moment of this construction* two light rays are emitted from the extremities of l in the direction of the two geodesics, they will follow the geodesics, irrespective of expansion, and meet in the origin under the angle $d\vartheta$. Thus (44) gives the observed angle, if R and l refer *to the moment of emission.* — This is the first of two important formulae, due to R. C. T o l m a n.

Again let E_0 be the energy emitted by a nebula during a suitably large unit of time. „Soon" after emission it will fill a spherical shell with thickness C (say). On observation, at angular distance χ, the thickness will have increased to CR_{obs}/R, if R and R_{obs} refer to the moments of emission and observation respectively. The surface of the shell in the moment of observation is $4\pi R_{obs}^2 \sin^2 \chi$, the energy, contained in it *then*, is $E_0 R/R_{obs}$. Hence the observed energy density ρ is

$$\rho = \frac{E_0}{4\pi C R_{obs}^4} \frac{R^2}{\sin^2 \chi}. \tag{45}$$

This is the second of the two important formulae due to T o l m a n.

H u b b l e and T o l m a n s paper, quoted above, gives a very careful analysis of how to compare (44) and (45) with observations in order to decide, whether the cause of the red-shift actually is expansion. The authors add a lucid and open-minded exposition of the present situation. The task is extremely intricate both from the observational and from the theoretical side. It is impossible to resume it in a few lines. In addition to all the rest of complexity, the general state of affairs in an *expanding* universe suggests, I think, the belief, that nebular diameters (l) and particularly nebular intrinsic luminosities (E_0) might very well themselves undergo, on the average, some kind of secular variation with R. If this possibility is envisaged, the hypothesis of expansion is probably easier to fit in with observations than any non-expansional explanation of the red-shift — although at first sight, i.e. with *constant* l and E_0, the reverse appears to be the case.

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2) R. C. Tolman, Relativity, Thermodynamics and Cosmology (Oxford, Clarendon Press, 1934) p. 371, equ. 149. 7.
3) E. Hubble and R. C. Tolman, Astrophysical Journal 82, 302, 1935; see formulae (3) and (4) there, which correspond to our (44) and (45) respectively.